

Quantum cobwebs: Universal entangling of quantum states

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Abstract

Entangling an unknown qubit with one type of reference state is generally impossible. However, entangling an unknown qubit with two types of reference states is possible. To achieve this, we introduce a new class of states called *zero sum amplitude* (ZSA) multipartite, pure entangled states for qubits and study their salient features. Using shared-ZSA state, local operation and classical communication we give a protocol for creating multipartite entangled states of an unknown quantum state with two types of reference states at remote places. This provides a way of encoding an unknown pure qubit state into a multiqubit entangled state. We quantify the amount of classical and quantum resources required to create universal entangled states. This is possibly a strongest form of quantum bit hiding with multiparties.

I. INTRODUCTION

In recent years we have learnt about what we can do and what we cannot do with the largely inaccessible information content of an unknown quantum state [1]. On the one hand linearity and unitarity of quantum theory are guiding principles and on the other hand they put several fundamental limitations on quantum information. Some of these limitations are no-cloning [2,3], no-deleting against a copy [4], and no-complementing [5]. Though exact operations are not allowed, these impossible operations can be made possible with a fidelity at least equal to that of the state estimation fidelity [6]. Processing of the vast amount of information contained in an unknown quantum state without destroying the coherence is an important task, in general.

Another key feature of the quantum world is the entangled nature of quantum states. Though a complete understanding of quantum entanglement is still lacking, many important developments have taken place in recent years in qualifying and quantifying multiparticle quantum entanglement [7]. Quantum entanglement is generally regarded as a very useful resource for quantum information processing [8]. In a striking discovery it was shown that without violating no-cloning principle an *unknown* quantum state can be teleported [9] with unit fidelity from one place to another using a quantum entangled channel and sending

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two bits of classical information. This has been also demonstrated experimentally [10–12]. Quantum entanglement can be used for dense coding [13], entanglement swapping [14,15], remote state preparation of special qubits [16], study of communication cost [17] and remote state preparation of arbitrary quantum states [18], teleportation of a unitary operator (quantum remote control) [19], telecloning [20], remote information concentration [21] and many more important tasks. In addition, it is a hope that entanglement will play a key role in quantum computation in giving its extra power compared to classical computers [22].

Here, we find yet another startling application of quantum entanglement. Imagine that we need to distribute an unknown qubit to more than one party. If we could distribute to many parties without entangling (i.e. in product states) then that would violate no-cloning principle. This means that the distributed state of a qubit with N parties must be in an entangled state. But creating an universal entangled state of an unknown state was shown to be impossible [24], i.e., starting with a state $|\psi\rangle_1|0\rangle_2\cdots|0\rangle_N$ we cannot create a symmetric universal entangled state $|\psi\rangle_1|0\rangle_2\cdots|0\rangle_N + |0\rangle_1|\psi\rangle_2\cdots|0\rangle_N + \cdots |0\rangle_1|0\rangle_2\cdots|\psi\rangle_N$. However, the surprising fact is that if we drop symmetric requirement then it is possible to create *two types of multiparticle entangled state of an unknown state*. Precisely, we are looking for a protocol where the resulting state will be either $a_1|\psi\rangle_1|0\rangle_2\cdots|0\rangle_N + a_2|0\rangle_1|\psi\rangle_2|0\rangle_3\cdots|0\rangle_N + \cdots a_N|0\rangle_1|0\rangle_2\cdots|\psi\rangle_N$ or $a_1|\psi\rangle_1|1\rangle_2\cdots|1\rangle_N + a_2|1\rangle_1|\psi\rangle_2|1\rangle_3\cdots|1\rangle_N + \cdots a_N|1\rangle_1|1\rangle_2\cdots|\psi\rangle_N$, where an unknown state $|\psi\rangle$ is entangled with reference states $|0\rangle$ and $|1\rangle$. The protocol we present here differs from the original aim of the universal entangler as proposed in [24] because in our scheme one can end up with two different type of universal entangled states. Further, Buzek and Hillery [24] have studied approximate methods to generate universal entangled states whereas we propose how to create exact universal entangled states. This is an important step in the sense that our protocol can produce universal entangled states (which people have previously thought to be impossible) and which works with unit probability of success. It is a hope that exact universal entangled states will play important roles for storage of quantum information against environmental decoherence [25].

In this paper we introduce a class of entangled states called *zero sum amplitude* (ZSA) entangled states which may have merit on their own. We present a protocol where upon using a special class of ZSA multipartite (say N , where N is the number of parties) shared entangled states, local operations and classical communication (LOCC), one can create two types of shared-entangled state of an unknown quantum state with $(N - 1)$ qubits at remote places. The information about an unknown state is distributed with all the N parties concerned *in a non-local way*. Thus, remote shared-entangling of an unknown state with multiparties is a very secure way to preserve the information about an unknown state (as long as N parties can maintain their quantum correlation). The present scheme could have some potential application in multiparty quantum cryptographic protocols which will be reported elsewhere [23].

The organisation of the present paper is as follows. In section 2, we introduce the zero sum amplitude entangled state and discuss its salient features. In section 3, we give our protocol for creating universal entangled states using a tripartite entangled state and quantify the resource needed to do the task. Further, in section 4 we generalise it to N -partite entangled states and quantify the amount of bipartite entanglement and classical communication needed to create $(N - 1)$ -partite quantum cobweb. We also explain why a classical correlated channel cannot be used to create universal entangled states. Our example

of N -partite zero sum amplitude entangled state (amplitudes being N th root of unity) shows that the amount of bipartite splitting entanglement goes as $1/N$ in the large N limit and the conclusion follows in section 5.

II. ZERO SUM AMPLITUDE ENTANGLED STATES FOR MULTIQUBITS

For the sake of generality, we introduce an arbitrary pure N -qubit zero sum amplitude (ZSA) entangled state $|\Phi\rangle_{12\dots N} \in \mathcal{H}^2 \otimes \dots \otimes \mathcal{H}^2$ (N times) given by

$$|\Phi\rangle_{12\dots N} = \sum_{i=1}^{2^N} c_i |i\rangle_{12\dots N}, \quad (1)$$

where $\{|i\rangle\}$ is an orthonormal basis for 2^N -dimensional Hilbert space, $\sum_{i=1}^{2^N} c_i = 0$ (i.e., all the complex amplitudes sum to zero) and $\sum_{i=1}^{2^N} |c_i|^2 = 1$ (i.e., the state is normalised to unity). The state space of a quantum system is the complex projective Hilbert space $\mathcal{P} = \mathcal{H}/U(1)$ which can be defined as a set of rays of the Hilbert space under the projection map $\Pi : \mathcal{H} \rightarrow \mathcal{P}$. The complex projective Hilbert space has one dimension less, i.e., $\dim \mathcal{P} = \dim \mathcal{H} - 1$. For a general ZSA state the dimension of the state space (viewed as a real manifold) is $D = (2^{N+1} - 3)$ (i.e. a D -dimensional real space) and requires D real parameters to specify the point on the quantum state space.

In the rest of the paper, we will consider a special class of ZSA states where the number of complex amplitudes is equal to the number of parties (and since each party possesses one qubit it is also equal to the total number of qubits) involved and N orthonormal states contain all zeros except at a single entry which contains a one. For example, a N -partite ZSA state is given by

$$|\Psi\rangle_{123\dots N} = c_1 |100\dots 0\rangle_{123\dots N} + c_2 |010\dots 0\rangle_{123\dots N} + \dots + c_N |00\dots 1\rangle_{123\dots N} = \sum_{k=1}^N c_k |x_k\rangle_{123\dots N}, \quad (2)$$

where $|x_k\rangle$ ($k = 1, 2, \dots, N$) is a N -bit string containing all 0's except that k th party contains 1 and the amplitudes obey ZSA condition $\sum_k c_k = 0$ and the normalisation $\sum_k |c_k|^2 = 1$. These class of states can be completely specified by $(2N - 3)$ real parameters.

To appreciate the remarkable features of these class of states we first discuss the case of two parties. When the number of parties is two, the ZSA state is given by

$$|\Psi\rangle_{12} = c_1 |10\rangle_{12} + c_2 |01\rangle_{12}. \quad (3)$$

The ZSA and normalisation conditions guarantee that the above state is nothing but an EPR singlet state $|\Psi^-\rangle_{12} = \frac{1}{\sqrt{2}}(|10\rangle_{12} - |01\rangle_{12})$, which is just one member of the Bell-states. This state is known to be locally equivalent to other Bell-states and can be used for successful quantum teleportation of an unknown qubit [27]. However throughout the paper whenever we mention multiparticle state we will consider three or more qubits, i.e., $N \geq 3$.

Let us introduce a class of tripartite zero sum amplitude normalised entangled state of qubits $|\Psi\rangle_{123} \in \mathcal{H}^2 \otimes \mathcal{H}^2 \otimes \mathcal{H}^2$ given by

$$|\Psi\rangle_{123} = c_1|100\rangle_{123} + c_2|010\rangle_{123} + c_3|001\rangle_{123}, \quad (4)$$

where c_i 's ($i = 1, 2, 3$) are non-zero amplitudes of the basis states where the i th qubit is in the state $|1\rangle$. These amplitudes obey $\sum_{i=1}^3 c_i = 0$ and $\sum_{i=1}^3 |c_i|^2 = 1$. Interestingly, states of the type (4), but without ZSA condition, have shown up in a variety of places in the literature [26–29]. In particular, Coffman *et al* [26] have shown that these states minimise the residual three-tangle. They also show up in the work of Brun and Cohen [27,28] on GHZ distillation and has been named as ‘triple’ states. Remarkably, Dür *et al* [29] have shown that any tripartite entangled state can be converted either to the GHZ class or W class of states by LOCC in a probabilistic manner. The only difference between the class of tripartite entangled states introduced here and the ones studied in [26,28,27,29] is that here the amplitudes obey the ZSA condition. We conjecture that the ZSA states are not locally equivalent to $|GHZ\rangle_{123\dots N} = 1/\sqrt{2}(|000\dots 0\rangle_{123\dots N} + |111\dots 1\rangle_{123\dots N})$ and $|W\rangle_{123\dots N} = 1/\sqrt{N} \sum_k |x_k\rangle_{123\dots N}$ states, except for the trivial case $N = 2$. Following the discovery of Dür *et al* [29] we can say that ZSA state may be related to W-states under stochastic LOCC. Also one can give a simple proof that the ZSA state is inequivalent to $|W\rangle_{123\dots N}$ and $|GHZ\rangle_{123\dots N}$ states under local unitary operations [30].

Note that this tripartite entangled state is not a maximally entangled state. The ZSA states are not maximally fragile [31], i.e., measurement of any one of the subsystems does not necessarily destroy the entanglement between remaining qubits. For example, if we project the first qubit onto computational basis $|0\rangle$, the state of the particles 2 and 3 is $c_2|10\rangle_{23} + c_3|01\rangle_{23}$, which is a non-maximally entangled state. But projection onto a basis $|1\rangle$ gives a disentangled state. This property holds with respect to all other qubits. The one particle reduced density matrix $\rho_k \in \mathcal{H}^2$ for any one of the three particles is not completely random but a pseudo-pure state given by

$$\rho_k = |c_k|^2 I + (1 - 2|c_k|^2)|0\rangle\langle 0|, k = 1, 2, 3. \quad (5)$$

Further, if we trace out Alice’s qubit, the two-qubit state at Bob and Charlie’s place is a *mixed entangled* state given by

$$\begin{aligned} \rho_{23} = & |c_1|^2 |0\rangle\langle 0| \otimes |0\rangle\langle 0| + |c_2|^2 |1\rangle\langle 1| \otimes |0\rangle\langle 0| + |c_3|^2 |0\rangle\langle 0| \otimes \\ & |1\rangle\langle 1| + c_2 c_3^* |1\rangle\langle 0| \otimes |0\rangle\langle 1| + c_2^* c_3 |0\rangle\langle 1| \otimes |1\rangle\langle 0|. \end{aligned} \quad (6)$$

That (6) is inseparable can be seen by applying Peres-Horodecki criterion [32,33] which is a necessary and sufficient one in $\mathcal{H}^2 \otimes \mathcal{H}^2$. This says that if a density matrix ρ is separable then the partial transpose has only nonnegative eigenvalues. If T is a transposition on space of bounded operators $\mathcal{B}(\mathcal{H})$, then the partial transpose PT (with respect to second subsystem) on $\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H})$ is defined as $\rho_{m\mu, n\nu}^{\text{PT}} = \rho_{m\nu, n\mu}$. Thus, the partial transpose of two-qubit density matrix (6) is given by

$$\rho_{23}^{\text{PT}} = \begin{pmatrix} |c_1|^2 & 0 & 0 & c_2^* c_3 \\ 0 & |c_3|^2 & 0 & 0 \\ 0 & 0 & |c_2|^2 & 0 \\ c_2 c_3^* & 0 & 0 & 0 \end{pmatrix}. \quad (7)$$

The eigenvalues of ρ_{23}^{PT} are $\lambda_1 = |c_2|^2$, $\lambda_2 = |c_3|^2$, $\lambda_3 = \frac{1}{2}(|c_1|^2 + \sqrt{|c_1|^4 + 4|c_2|^2|c_3|^2})$ and $\lambda_4 = \frac{1}{2}(|c_1|^2 - \sqrt{|c_1|^4 + 4|c_2|^2|c_3|^2})$. Though first three eigenvalues are nonnegative the last

one is not (one can check that $\lambda_3\lambda_4$ is a negative number). Therefore, the two qubit density matrix ρ_{23} is inseparable. The same is true if we trace out any other qubit and look at the density matrix of the two-qubit system. A particular measure of entanglement for mixed state is ‘entanglement of formation’ [34,35]. The entanglement of formation of ρ_{23} can be computed explicitly and it is given by

$$E_{23} = -\frac{1}{2}(1 + \sqrt{1 - 4|c_2|^2|c_3|^2}) \log\left[\frac{1}{2}(1 + \sqrt{1 - 4|c_2|^2|c_3|^2})\right] - \frac{1}{2}(1 - \sqrt{1 - 4|c_2|^2|c_3|^2}) \log\left[\frac{1}{2}(1 - \sqrt{1 - 4|c_2|^2|c_3|^2})\right]. \quad (8)$$

We will see in the section 3. that the mixed state density matrix (6) is transformed to a pure state with LOCC. Next we come to the main result of our paper.

III. UNIVERSAL ENTANGLING OF UNKNOWN QUBIT WITH TWO PARTIES

In what follows we give our protocol for creating two types of universal entangled states of an unknown state at two remote locations. Suppose Alice, Bob and Charlie at remote locations share an entangled state (4) and have access to particles 1, 2 and 3, respectively. An unknown qubit is given to Alice in the form

$$|\psi\rangle_a = \alpha|0\rangle_a + \beta|1\rangle_a, \quad (9)$$

where $\alpha = \cos \frac{\theta}{2}$ and $\beta = \sin \frac{\theta}{2} \exp(i\phi)$. We show that Alice can always *create an entangled state of any unknown state with a reference state* $|0\rangle$ or $|1\rangle$ shared between Bob and Charlie by sending two bits of information to both of them. The combined state of the input and the tripartite ZSA entangled state $|\psi\rangle_a \otimes |\Psi\rangle_{123}$ can be expressed in terms of Bell-states [36] of particle a and 1 as

$$\begin{aligned} |\psi\rangle_a \otimes |\Psi\rangle_{123} = \frac{1}{\sqrt{2}} & \left[|\Phi^+\rangle_{a1} \otimes (c_2\alpha|10\rangle_{23} + c_3\alpha|01\rangle_{23} + \right. \\ & c_1\beta|00\rangle_{23}) + |\Phi^-\rangle_{a1} \otimes (c_2\alpha|10\rangle_{23} + c_3\alpha|01\rangle_{23} - c_1\beta|00\rangle_{23}) \\ & + |\Psi^+\rangle_{a1} \otimes (c_1\alpha|00\rangle_{23} + c_2\beta|10\rangle_{23} + c_3\beta|01\rangle_{23}) + \\ & \left. |\Psi^-\rangle_{a1} \otimes (c_1\alpha|00\rangle_{23} + c_2\beta|10\rangle_{23} - c_3\beta|01\rangle_{23}) \right]. \quad (10) \end{aligned}$$

Using the zero sum amplitude property, i.e., $\sum_i c_i = 0$, we can rewrite the combined state as

$$\begin{aligned} |\psi\rangle_a \otimes |\Psi\rangle_{123} = \frac{1}{\sqrt{2}} & \left[|\Phi^+\rangle_{a1} \otimes (c_2 i\sigma_y |\psi\rangle_2 |0\rangle_3 + c_3 |0\rangle i\sigma_y |\psi\rangle_3) + \right. \\ & |\Phi^-\rangle_{a1} \otimes (c_2 \sigma_x |\psi\rangle_2 |0\rangle_3 + c_3 |0\rangle \sigma_x |\psi\rangle_3) - |\Psi^+\rangle_{a1} \otimes (c_2 \sigma_z |\psi\rangle_2 |0\rangle_3 \\ & \left. + c_3 |0\rangle \sigma_z |\psi\rangle_3) + |\Psi^-\rangle_{a1} \otimes (c_2 |\psi\rangle_2 |0\rangle_3 + c_3 |0\rangle |\psi\rangle_3) \right], \quad (11) \end{aligned}$$

where σ_x, σ_y and σ_z are Pauli matrices. Now Alice performs a joint measurement on particles a and 1. Since a Bell-basis measurement will give four possible outcomes $\{|\Phi^\pm\rangle, |\Psi^\pm\rangle\}$ she can get two classical bits of information. Then, Alice sends two classical bits to Bob and

Charlie both, who in turn can apply certain local unitary operations to share an entangled state of an unknown state with reference states such as $|0\rangle$ or $|1\rangle$. For example, if the outcome is $|\Phi^+\rangle$ or $|\Phi^-\rangle$, then after receiving classical information Bob and Charlie will apply $i\sigma_y \otimes i\sigma_y$ or $\sigma_x \otimes \sigma_x$, respectively. They will be sharing an entangled state given by

$$|\psi^{(1)}\rangle_{23} = c_2|\psi\rangle_2|1\rangle_3 + c_3|1\rangle_2|\psi\rangle_3, \quad (12)$$

where $|\psi^{(1)}\rangle_{23}$ is an *universal entangled state* of an unknown qubit with a reference state $|1\rangle$. This is universal because the protocol works perfectly for any input qubit $|\psi\rangle$. If the result is $|\Psi^+\rangle$ or $|\Psi^-\rangle$ then, after receiving classical communication Bob and Charlie will apply $\sigma_z \otimes \sigma_z$ or $I \otimes I$, respectively. In this case they will be sharing an entangled state given by

$$|\psi^{(0)}\rangle_{23} = c_2|\psi\rangle_2|0\rangle_3 + c_3|0\rangle_2|\psi\rangle_3, \quad (13)$$

where the states $|\psi^{(0)}\rangle_{23}$ is an *universal entangled state* of an unknown state with a reference state $|0\rangle$. For successful creation of universal entangled states $|\psi^{(0)}\rangle_{23}$ or $|\psi^{(1)}\rangle_{23}$ two classical bits are needed from Alice. Note that the states in equations (12) and (13) are not normalised. The normalisation constant for (12) is $N(\beta) = 1/\sqrt{|c_2|^2 + |c_3|^2 + 2|\beta|^2\text{Re}(c_2^*c_3)}$ and for (13) is $N(\alpha)$, where $N(\alpha)$ can be obtained from $N(\beta)$ by replacing β with α .

An interesting observation is that if the state $|\psi\rangle$ is in a known state such as $|0\rangle$ or $|1\rangle$, then one may use this scheme for quantum cryptographic purposes. For example, if $|\psi\rangle = |0\rangle$, then $|\psi^{(0)}\rangle_{23}$ is not an entangled but $|\psi^{(1)}\rangle_{23}$ is. Similarly, if $|\psi\rangle = |1\rangle$, then $|\psi^{(1)}\rangle_{23}$ is not entangled but $|\psi^{(0)}\rangle_{23}$ is. This may provide a way to generate a coded message (detailed discussions are beyond the scope of the present paper and the results will be reported elsewhere [23]).

IV. CREATING QUANTUM COBWEBS

The states that we have created by this protocol are very special. One can check that there is no local unitary operation $\mathcal{H}_2 \otimes \mathcal{H}_3$ that can disentangle the unknown state perfectly. Even if both parties come together and perform joint unitary and measurement operations they cannot disentangle the qubit perfectly. Since a general quantum operation is a positive, linear, trace preserving map that has a unitary representation involving ancilla, let us assume that there is a unitary operator that disentangles any arbitrary qubit perfectly. The action of the unitary operator on universal entangled state of $|\psi\rangle$ and $|\bar{\psi}\rangle = \alpha|1\rangle - \beta^*|0\rangle$ (with $\langle\psi|\bar{\psi}\rangle = 0$) will be given by

$$\begin{aligned} N(\alpha)(c_2|\psi\rangle_2|0\rangle_3 + c_3|0\rangle_2|\psi\rangle_3)|A\rangle &\rightarrow |0\rangle|\psi\rangle|A'\rangle, \\ N(\beta)(c_2|\bar{\psi}\rangle_2|0\rangle_3 + c_3|0\rangle_2|\bar{\psi}\rangle_3)|A\rangle &\rightarrow |0\rangle|\bar{\psi}\rangle|A''\rangle, \end{aligned} \quad (14)$$

where $|A\rangle$ is the initial and $|A'\rangle, |A''\rangle$ are the final states of the ancilla, $N(\alpha)$ and $N(\beta)$ are normalisation constants for entangled states of $|\psi^{(0)}\rangle$ and $|\bar{\psi}^{(0)}\rangle$, respectively. Taking the inner product we have $2N(\alpha)N(\beta)\alpha\beta^*\text{Re}(c_2c_3^*) = 0$ and this can never be satisfied for any non-zero values of c_2, c_3, α and β . Therefore, we cannot disentangle the state even by joint action and irreversible operation. Thus, the unknown state (containing some secret information) can remain simultaneously with two parties in a non-local manner. The element

of surprise is in the fact that this seems to be an irreversible conversion process (somewhat analogous to other irreversible distillation process.) This class of states we call *quantum cobwebs* because once they are created the unknown state is trapped inside the multiparticle entangled state. Though this feature may look undesirable to some readers it is indeed very useful in quantum cryptographic schemes. Often, new quantum information processing protocols are double-edged swords. If there is a negative aspect of a protocol there is a great positive aspect as well. However, here we would like to leave it as an open question whether these universal entangled states can be disentangled perfectly using entanglement assisted local operation and classical communication.

As mentioned in the introduction, recently it was shown [24] that there is no unitary operator which can create a perfect symmetric universal entangled state that will take $|\psi\rangle_1|0\rangle_2 \rightarrow |\psi\rangle_1|0\rangle_2 + |0\rangle_1|\psi\rangle_2$. Similarly, the reverse operation, i.e., a perfect disentan- gler is also not possible [42]. But in our protocol we have circumvented this limitation and achieved two types of arbitrary *universal perfect entanglers* with unit probability using shared entanglement, local operations and classical communications (LOCC). However, our universal entangled states are not permutationally invariant. Of course, our scheme may not be the only way to generate universal entangled states. It could be possible to consider a unitary operation on an unknown state along with ancillas and one may be able to create two types of universal entangled states with a postselection of measurement result. But this needs further investigation.

It may be worth mentioning that if Bob and Charlie perform non-local unitary operation and measurement, then one of them can recover the state with unit fidelity in a *probabilistic manner*. For example, to disentangle $|\psi^{(0)}\rangle_{23}$ Bob and Charlie can come together and perform a CNOT operation followed by a measurement of particle 2 in the basis $\{|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\}$. When Bob gets $|+\rangle$, Charlie's qubit is in the state $|\psi\rangle$ and when Bob gets $|-\rangle$ Charlie's qubit is not in the state $|\psi\rangle$, (i.e. there is an error in getting $|\psi\rangle$) so they can discard this. The probability of success is $P = [|c_2|^2 + |c_3|^3 + 2\text{Re}(c_2^*c_3)]/2[|c_2|^2 + |c_3|^3 + 2\alpha^2\text{Re}(c_2^*c_3)]$ which is greater than half (i.e. better than a random guess).

V. RESOURCE FOR UNIVERSAL ENTANGLED STATES

Next, we quantify the amount of nonlocal quantum resource needed to create a remote shared-entangled state. In general quantifying amount of entanglement in multiparticle system is still a difficult problem [37]. Moreover, for more than two parties there is no unique measure of quantum entanglement [38]. However, our purpose is not to provide a measure of entanglement for ZSA sates. We simply observe the following points. The tripartite system can be partitioned in three different ways, i.e A vs BC , B vs AC and C vs AB , there are three different ways of calculating the bipartite entanglement. Since we are interested to know the quantum resources between A versus BC (as we are creating universal entanglers for BC), we look at the amount of bipartite entanglement with respect to splitting of particles A vs BC . This is given by the von Neumann entropy of the reduced density matrix ρ_1 [39]

$$\begin{aligned} E(A \text{ vs } BC) &= E(\rho_1) = -\text{tr}(\rho_1 \log \rho_1) = \\ &= -(1 - |c_1|^2) \log(1 - |c_1|^2) - |c_1|^2 \log |c_1|^2. \end{aligned} \quad (15)$$

Though, the use of von Neumann entropy as a measure of entanglement for bipartite system is justified when asymptotically large number of copies are involved [40], we use this for its simplicity. Therefore, we can roughly say that using $E(\rho_1)$ amount of entanglement and communication of two classical bits to Bob and Charlie one can create two types of quantum universal entanglers for an unknown state. Thus a mixed entangled state (6) is converted to a pure universal entangled state after receiving classical communication from Alice. (Recall a similar situation in quantum teleportation, where a completely random mixture is converted to a pure unknown state). This is very interesting process where a mixed entangled state shared between two parties Bob and Charlie is purified to a pure entangled state using LOCC along with assisted LOCC from a third party Alice.

We can also argue that no classical correlated state (CCS) can create an universal entangled state of an unknown state. If we could create an universal entangled state using CCS via local operation and classical communication then we could create some amount of entanglement between Bob and Charlie. But we know that via LOCC one cannot create any entanglement [38], hence CCS cannot create any universal entangled states.

We can actually quantify the amount of entanglement present in a bipartite universal entangled state (quantum cobweb). When the universal entangled state is of the type (13), then the reduced density matrix of qubit 2 at Bob's place is

$$\rho_2 = N(\alpha)^2[|c_2|^2|\psi\rangle\langle\psi| + |c_3|^2|0\rangle\langle 0| + c_2c_3^*\alpha|\psi\rangle\langle 0| + c_2^*c_3\alpha|0\rangle\langle\psi|]. \quad (16)$$

In the bipartite case the Schmidt decomposition theorem [41] guarantees that the eigenvalues of the reduced density matrices of B and C will be identical. They are given by $\eta_{\pm} = \frac{1}{2}(1 \pm \sqrt{1 - 4\epsilon})$, where $\epsilon = 4N(\alpha)^4|\beta|^4|c_2|^2|c_3|^2$. Therefore, the amount of entanglement will be

$$E(|\psi^{(0)}\rangle_{23}) = -\text{tr}(\rho_2 \log \rho_2) = -\text{tr}(\rho_3 \log \rho_3) = -\eta_+ \log \eta_+ - \eta_- \log \eta_-. \quad (17)$$

As an example, if the amplitudes are cube roots of unity, then the zero sum amplitude entangled state is of the form

$$|\Psi\rangle_{123} = \frac{1}{\sqrt{3}}(|100\rangle_{123} + e^{2\pi i/3}|010\rangle_{123} + e^{-2\pi i/3}|001\rangle_{123}). \quad (18)$$

The reduced density matrices for each of the subsystem are same and also has equal spectrum. It is given by $\rho_1 = \rho_2 = \rho_3 = \text{diag}(2/3, 1/3)$. Therefore, the amount of bipartite entanglement between any partitioning is $E(\rho_1) = E(\rho_2) = E(\rho_3) = 1 - (5 - 3 \log 3)/3 = .9\text{ebits}$. Thus, with a use of .9 ebits of entanglement and two cbits of communication one can create, for example, a universal entangled state of the form

$$|\Psi^{(0)}\rangle_{23} = \frac{1}{\sqrt{3}}(e^{2\pi i/3}|\psi\rangle_2|0\rangle_3 + e^{-2\pi i/3}|0\rangle_2|\psi\rangle_3). \quad (19)$$

VI. UNIVERSAL ENTANGLED STATE FOR MULTIPARTIES

We can generalise the universal quantum entangler for $(N-1)$ parties where an unknown qubit can be entangled with a reference state and shared with $(N-1)$ parties. Let there

be N parties in a network of N nodes each having access to a single qubit. They share N -partite zero sum amplitude entangled state $|\Psi\rangle_{123\dots N} \in \mathcal{H}^{2^{\otimes N}}$ given by (2).

Now, we describe how Alice can create an $(N-1)$ -partite entangled state of any unknown state with a reference state $|0\rangle$ or $|1\rangle$ shared between Bob, Charlie....and Nancy by sending two bits of information to the concerned parties. The combined state of the unknown qubit and N -partite entangled state $|\psi\rangle_a \otimes |\Psi\rangle_{123\dots N}$ can be expressed in terms of Bell-states of particle a and 1 as (again using the zero sum amplitude property)

$$\begin{aligned} |\psi\rangle_a \otimes |\Psi\rangle_{123\dots N} = & \frac{1}{\sqrt{2}} \left[|\Phi^+\rangle_{a1} \otimes \sum_{k=2}^N c_k |(i\sigma_y)\psi_{(k)}^{(0)}\rangle_{23\dots N} + \right. \\ & |\Phi^-\rangle_{a1} \otimes \sum_{k=2}^N c_k |(\sigma_x)\psi_{(k)}^{(0)}\rangle_{23\dots N} - |\Psi^+\rangle_{a1} \otimes \sum_{k=2}^N c_k |(\sigma_z)\psi_{(k)}^{(0)}\rangle_{23\dots N} + \\ & \left. |\Psi^-\rangle_{a1} \otimes \sum_{k=2}^N c_k |\psi_{(k)}^{(0)}\rangle_{23\dots N} \right], \end{aligned} \quad (20)$$

where $|\psi_{(k)}^{(0)}\rangle_{23\dots N} = |0\rangle_2|0\rangle_3 \cdots |\psi\rangle_k \cdots |0\rangle_N$ is a $(N-1)$ qubit strings containing all qubits in the state $|0\rangle$ except that the k th party contains the unknown state $|\psi\rangle$. Alice performs a joint Bell-state measurement on particles a and 1. If the outcome is $|\Phi^+\rangle$ or $|\Phi^-\rangle$ then after sending classical communication to the concerned $(N-1)$ parties, they will apply $i\sigma_y \otimes \cdots \otimes i\sigma_y$ or $\sigma_x \otimes \cdots \otimes \sigma_x$, respectively. They will end up sharing an entangled state given by

$$|\psi^{(1)}\rangle_{23\dots N} = \sum_{k=2}^N c_k |\psi_{(k)}^{(1)}\rangle_{23\dots N}, \quad (21)$$

where $|\psi_{(k)}^{(1)}\rangle_{23\dots N} = |1\rangle_2|1\rangle_3 \cdots |\psi\rangle_k \cdots |1\rangle_N$ is a $(N-1)$ qubit strings that contains all qubits in the state $|1\rangle$ except that the k th party contains the unknown state $|\psi\rangle$. If the outcome is $|\Psi^+\rangle$ or $|\Psi^-\rangle$ then after receiving classical communication, $(N-1)$ parties will apply $\sigma_z \otimes \cdots \otimes \sigma_z$ or $I \otimes \cdots \otimes I$ (do nothing), respectively. Thus, they will end up sharing an entangled state given by

$$|\psi^{(0)}\rangle_{23\dots N} = \sum_{k=2}^N c_k |\psi_{(k)}^{(0)}\rangle_{23\dots N}. \quad (22)$$

Thus, with the use of zero sum amplitude entangled state and two classical bits one can create universal entangled states of an unknown state with two types of reference states that have been shared between $(N-1)$ parties at remote locations. Thus, $|\psi^{(0)}\rangle_{23\dots N}$ and $|\psi^{(1)}\rangle_{23\dots N}$ are $(N-1)$ node quantum cobwebs from which an unknown state cannot be disentangled perfectly by local or nonlocal unitary operations. The multiparty cobweb states in (21) and (22) are not normalised as expected.

The reduced density matrices for single qubits (after tracing out other $(N-1)$ qubits from the N -partite state (2)) is given by (5) with $k = 1, 2, \dots, N$. With N parties there are $N(N-1)/2$ possible bipartite entanglements but we are interested in entanglement with respect to N number of bipartite partitioning (where we make partitioning of one qubit versus all other qubits). The amount of bipartite entanglement with respect to splitting between

qubit 1 vs $(N - 1)$ is again given by (15). Therefore, with the use of $E(\rho_1)$ amount of entanglement and two classical bits to $(N - 1)$ parties one can create two types of $(N - 1)$ partite universal entangled states.

As an example, if the amplitudes are N th roots of unity, then the zero sum amplitude entangled state is given by

$$|\Psi\rangle_{123\dots N} = \frac{1}{\sqrt{N}} \sum_{k=1}^N e^{i2\pi k/N} |x_k\rangle_{123\dots N}. \quad (23)$$

The reduced density matrix of any of the qubit is identical and is given by $\rho_k = \text{diag}[(1 - 1/N), 1/N]$. Therefore, the amount of bipartite entanglement is independent of the choice of N possible bipartite partitioning. The bipartite entanglement E with respect to splitting between particle 1 and the rest $(N - 1)$ qubits is

$$E = E(\rho_1) = -[(1 - \frac{1}{N}) \log(1 - \frac{1}{N}) + \frac{1}{N} \log \frac{1}{N}] \quad (24)$$

With the use of E bits of entanglement one can create an universal entangled state of $(N - 1)$ qubits as

$$|\psi^{(0)}\rangle_{23\dots N} = \frac{1}{\sqrt{N}} \sum_{k=2}^N e^{i2\pi k/N} |\psi_{(k)}^{(0)}\rangle_{23\dots N}. \quad (25)$$

If the number of parties N becomes very large $E \rightarrow 1/N$, this approaches zero i.e., the bipartite entanglement for the state (23) cannot be unlimitedly distributed between large number of parties. For large but finite N we can say that with the use of $O(1/N)$ ebits of bipartite entanglement we can prepare a universal entangled state (25) for an unknown state with $O(N)$ parties at remote locations. In a different context, it was shown [45] that bipartite entanglement distributed between N parties goes as $2/N$.

VII. CONCLUSION

In this paper, we have introduced a class of zero sum amplitude multipartite entangled states and studied their properties. Interestingly, when the number of parties is two, the ZSA entangled state is exactly an EPR state. We have presented a protocol where one can create two types of universal entangled states of an unknown state with reference states using shared ZSA entangled states and LOCC, which was thought to be an impossible task. This class of states may be called quantum cobwebs. This surprising feature exploits one property that is the zero sum amplitude nature of the original shared entangled state between N parties. Creating a quantum cobweb could have some strategic applications, where some secret information is shared with every body but no one can salvage that information. This is very useful for cryptographic schemes. It may be remarked that though the original quantum teleportation uses maximally bipartite entangled states, one can also use three particle and four particle GHZ states for quantum teleportation [43,44]. Ours is one example, where these class of multipartite states are pure entangled states but *are not useful* for quantum teleportation. We hope that the nature of ZSA and universal entangled states will throw

some new light on the nature of quantum information and role of entanglement. Some open questions include: Is our scheme the most simple scheme for creating cobwebs? Why the ZSA states are so special and can one create cobwebs with other entangled states? In future, one can also explore if these multipartite ZSA entangled states can be employed for some other quantum information processing tasks.

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REFERENCES

- [1] C. A. Caves and C. A. Fuchs, “Quantum Information: How much information in a state vector?”, *The Dilemma of Einstein, Podolsky and Rosen-60 Years Later*, Edited by A. Mann and M. Revzen, Ann. Israel Phys. Soc. **12**, 226 (1996).
- [2] W. K. Wootters and W. H. Zurek, Nature **299**, 802 (1982).
- [3] D. Dieks, Phys. Lett. A **92**, 271 (1982)
- [4] A. K. Pati and S. L. Braunstein, Nature **404**, 164 (2000)
- [5] V. Buzek, M. Hillery and R. F. Werner, Phys. Rev. A **60**, R2626 (1999).
- [6] S. Massar and S. Popescu, Phys. Rev. Lett. **74**, 1259 (1995).
- [7] For example, see the latest thoughts in an article by D. Bruss, quant-ph/0110078 and references therein.
- [8] C. H. Bennett, H. J. Bernstein, S. Popescu and B. Schumacher, Phys. Rev. A **53**, 2046 (1996).
- [9] C. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).
- [10] D. Bouwmeester, J. W. Pan, K. Mattle, M. Eibl, H. Weinfurter, and A. Zeilinger, Nature **390**, 575 (1997).
- [11] D. Boschi, S. Branca, F. De Martini, L. Hardy and S. Popescu, Phys. Rev. Lett. **80**, 1121 (1998).
- [12] A. Furusawa, J. L. Sorensen, S. L. Braunstein, C. A. Fuchs, H. J. Kimble and E. S. Polzik, Science **282**, 706 (1998).
- [13] C. H. Bennett and S. Wiesner, Phys. Rev. Lett. **69**, 2881 (1992).
- [14] M. Zukowski, A. Zeilinger, M. A. Horne and E. Ekert, Phys. Rev. Lett. A **71**, 4278 (1993).
- [15] S. Bose, V. Vedral and P. Knight, Phys. Rev. A **57**, 822 (1998).
- [16] A. K. Pati, Phys. Rev. A **63**, 014320-1 (2001)
- [17] H. K. Lo, Phys. Rev. A **62**, 012313 (2000).
- [18] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, B. M. Terhal, and W. K. Wootters, Phys. Rev. Lett. **87**, 077902 (2001).
- [19] S. F. Huelga, J. A. Vaccaro, A. Chefless and M. B. Plenio, Phys. Rev. A **63**, 042303 (2001).
- [20] M. Murao *et al*, Phys. Rev. A **59**, 156 (1999).
- [21] M. Murao and V. Vedral, Phys. Rev. Lett. **86**, 352 (2001).
- [22] M. Nielsen and I. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press, Cambridge, UK, 2000.
- [23] A. K. Pati, *Quantum cryptography using zero sum amplitude entangled states* (in preparation).
- [24] V. Buzek and M. Hillery, Phys. Rev. A, **62**, 022303 (2000).
- [25] A. Barenco, A. Berthiaume, D. Deutsch, A. Ekert, R. Jozsa, and C. Macchiavello, *SIAM Journal of Computing* **26**, 1541 (1997).
- [26] V. Coffman, J. Kundu and W. K. Wootters, Phys. Rev. A **61** 052306 (2000).
- [27] O. Cohen and T. Brun, Phys. Rev. Lett. **84**, 5908 (2000).
- [28] T. Brun and O. Cohen, Phys. Lett. A **281**, 88 (2001).
- [29] W. Dür, G. Vidal and J. I. Cirac, Phys. Rev. A **62**, 062314-1 (2000)
- [30] Suppose, there is a local unitary operator $U = U_1 \otimes U_2 \cdots U_N$ such that $U|W\rangle_{123\dots N} =$

$|\Psi\rangle_{123\dots N}$. This implies that the amplitudes obey $c_l = 1/\sqrt{N} \sum_k U_{lk}$ and zero sum amplitude condition implies $\sum_{lk} U_{lk} = 0$. First, consider the case where the unitary operator is uni-local (unitary operator acting only on one party), i.e., $U = U_1 \otimes I_2 \cdots I_N$. The general case would be obtained by composing uni-local unitary operators $U_1 \otimes I_2 \cdots I_N$ with $I_1 \otimes U_2 \cdots I_N$ and so on [29]. Inserting U , we have the equation $\langle 1|U_1|1\rangle + (N-1)\langle 0|U_1|0\rangle = 0$. But this condition can never be satisfied by any local unitary operator. By composing uni-local unitary operators one can see that there is no arbitrary local unitary operator such that $|\Psi\rangle$ and $|W\rangle$ are locally equivalent. Similarly, one can show that $|\Psi\rangle$ and $|GHZ\rangle$ states are not related by local unitary operators for $N \geq 3$. If they are, then zero sum amplitude condition implies that $\langle 1|U_1|0\rangle + \langle 1|U_1|1\rangle = 0$ and this can never be satisfied. This proves that the ZSA state is not equivalent to $|W\rangle$ and $|GHZ\rangle$ states under local unitary operators.

- [31] N. Gisin and H. B. Pasquinucci, Phys. Lett. A **246**, 1 (1998).
- [32] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996).
- [33] M. Horodecki, P. Horodecki and R. Horodecki, Phys. Lett. A **223**, 1 (1996).
- [34] C. H. Bennett, D. P. DiVincenzo, J. Smolin, and W. K. Wootters, Phys. Rev. A **54**, 3824 (1996).
- [35] W. K. Wootters, quant-ph/ 9709029.
- [36] S. L. Braunstein, A. Mann and M. Reznik, Phys. Rev. Lett. **68**, 3259 (1992).
- [37] See for example, G. Vidal and R. F. Werner, quant-ph/0102117.
- [38] G. Vidal, W. Dür and J. I. Cirac, Phys. Rev. Lett. **85**, 658 (2000).
- [39] S. Popescu and D. Rohrlich, Phys. Rev. A **56**, R3319 (1997).
- [40] G. Vidal, Phys. Rev. Lett. **83**, 1046 (1999).
- [41] A. Ekert and P. Knight, Am. J. Phys. **63**, 415 (1995); A. K. Pati, Phys. Lett. A **278**, 118 (2000).
- [42] V. Buzek and M. Hillery, Phys. Rev. A, **62**, 052303 (2000).
- [43] A. Karlson and M. Bourennane, Phys. Rev. A **58**, 4394 (1998).
- [44] A. K. Pati, Phys. Rev. A, **61**, 022308 (2000).
- [45] M. Koashi, V. Buzek, and N. Imoto, Phys. Rev. A **62**, 050302(R) (2000).